

Asymptotic behavior of the eigenvalues of the $p(x)$ -Laplacian*

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Abstract

We obtain asymptotic estimates for the eigenvalues of the $p(x)$ -Laplacian defined consistently with a homogeneous notion of first eigenvalue recently introduced in the literature.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 1$ and let $p \in C(\overline{\Omega}, (1, \infty))$. The purpose of this paper is to study the asymptotic behavior of the eigenvalues of the problem

$$-\operatorname{div} \left(\left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \right) = \lambda S(u) \left| \frac{u}{k(u)} \right|^{p(x)-2} \frac{u}{k(u)}, \quad u \in W_0^{1,p(x)}(\Omega), \quad (1.1)$$

where

$$K(u) = \|\nabla u\|_{p(x)}, \quad k(u) = \|u\|_{p(x)}, \quad S(u) = \frac{\int_{\Omega} \left| \frac{\nabla u(x)}{K(u)} \right|^{p(x)} dx}{\int_{\Omega} \left| \frac{u(x)}{k(u)} \right|^{p(x)} dx}.$$

The equation in (1.1) was derived by Franzina and Lindqvist in [5] as the Euler-Lagrange equation arising from minimizing the Rayleigh quotient $K(u)/k(u)$ over $W_0^{1,p(x)}(\Omega) \setminus \{0\}$. It was shown there that the first eigenvalue $\lambda_1 > 0$ and has an associated eigenfunction $\varphi_1 > 0$.

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We recall that the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ consists of all measurable functions u on Ω with the Luxemburg norm

$$\|u\|_{p(x)} := \inf \left\{ \nu > 0 : \int_{\Omega} \left| \frac{u(x)}{\nu} \right|^{p(x)} \frac{dx}{p(x)} \leq 1 \right\} < \infty.$$

The Sobolev space $W^{1,p(x)}(\Omega)$ consists of functions $u \in L^{p(x)}(\Omega)$ with a distributional gradient $\nabla u \in L^{p(x)}(\Omega)$, and the norm in this space is $\|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. The space $W_0^{1,p(x)}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the above norm, and has the equivalent norm $\|\nabla u\|_{p(x)}$. We refer the reader to Diening et al. [2] for details on these spaces.

It was shown in [5] that

$$(K'(u), v) = \frac{\int_{\Omega} \left| \frac{\nabla u(x)}{K(u)} \right|^{p(x)-2} \frac{\nabla u(x)}{K(u)} \cdot \nabla v(x) dx}{\int_{\Omega} \left| \frac{\nabla u(x)}{K(u)} \right|^{p(x)} dx}, \quad u, v \in W_0^{1,p(x)}(\Omega)$$

and

$$(k'(u), v) = \frac{\int_{\Omega} \left| \frac{u(x)}{k(u)} \right|^{p(x)-2} \frac{u(x)}{k(u)} v(x) dx}{\int_{\Omega} \left| \frac{u(x)}{k(u)} \right|^{p(x)} dx}, \quad u, v \in W_0^{1,p(x)}(\Omega),$$

so the eigenvalues and eigenfunctions of (1.1) on the manifold $\mathcal{M} = \{u \in W_0^{1,p(x)}(\Omega) : k(u) = 1\}$ coincide with the critical values and critical points of $\tilde{K} := K|_{\mathcal{M}}$. In the next section we will show that \tilde{K} satisfies the (PS) condition, so we can define an increasing and unbounded sequence of eigenvalues of (1.1) by a minimax scheme. Although the standard scheme uses Krasnoselskii's genus, we prefer to use a cohomological index as in Perera [11] since this gives additional Morse theoretic information that is often useful in applications.

Let us recall the definition of the \mathbb{Z}_2 -cohomological index of Fadell and Rabinowitz [3]. Let \mathcal{F} denote the class of symmetric subsets of \mathcal{M} . For $M \in \mathcal{F}$, let $\overline{M} = M/\mathbb{Z}_2$ be the quotient space of M with each u and $-u$ identified, let $f : \overline{M} \rightarrow \mathbb{RP}^\infty$ be the classifying map of \overline{M} , and let $f^* : H^*(\mathbb{RP}^\infty) \rightarrow H^*(\overline{M})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. Then the cohomological index of M is defined by

$$i(M) = \begin{cases} \sup \{m \geq 1 : f^*(\omega^{m-1}) \neq 0\}, & M \neq \emptyset \\ 0, & M = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{RP}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{RP}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in \mathbb{R}^m , $m \geq 1$ is the inclusion $\mathbb{RP}^{m-1} \subset \mathbb{RP}^\infty$, which induces isomorphisms on H^q for $q \leq m-1$, so $i(S^{m-1}) = m$.

Set

$$\lambda_j := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq j}} \sup_{u \in M} \tilde{K}(u), \quad j \geq 1. \quad (1.2)$$

Then (λ_j) is a sequence of eigenvalues of (1.1) and $\lambda_j \nearrow \infty$. Moreover,

$$\lambda_j < \lambda \leq \lambda_{j+1} \implies i(\tilde{K}^\lambda) = j,$$

where $\tilde{K}^\lambda = \{u \in \mathcal{M} : \tilde{K}(u) < \lambda\}$, so

$$i(\tilde{K}^\lambda) = \#\{j : \lambda_j < \lambda\} \quad \forall \lambda \in \mathbb{R} \quad (1.3)$$

(see Propositions 3.52 and 3.53 of Perera et al. [12]). Our main result is the following.

Theorem 1.1. *If $1 < p^- \leq p(x) \leq p^+ < \infty$ for all $x \in \Omega$ and*

$$\sigma := n \left(\frac{1}{p^-} - \frac{1}{p^+} \right) < 1,$$

then there are constants $C_1, C_2 > 0$, that depend only on n and p^\pm , such that

$$C_1 |\Omega| \lambda^{n/(1+\sigma)} \leq \#\{j : \lambda_j < \lambda\} \leq C_2 |\Omega| \lambda^{n/(1-\sigma)} \quad \text{for } \lambda > 0 \text{ large,}$$

where $|\Omega|$ is the Lebesgue measure of Ω .

This result is a contribution towards understanding the spectrum of the $p(x)$ -Laplacian, which many researchers have recently found to be somewhat puzzling. For example, it is currently unknown if the first eigenvalue is simple, or if a given positive eigenfunction is automatically a first eigenfunction. Affirmative answers were given to both of these questions for the usual eigenvalue problem for the p -Laplacian,

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u, \quad u \in W_0^{1,p}(\Omega), \quad (1.4)$$

where $p > 1$ is a constant, in Lindqvist [8, 9] (see also [10]). It should be noted that, in the case of constant p , (1.1) reduces, not to the problem (1.4), which is homogeneous of degree $p-1$, but rather to the nonlocal problem

$$-\operatorname{div} \left(\frac{|\nabla u|^{p-2} \nabla u}{\|\nabla u\|_p^{p-1}} \right) = \lambda \frac{|u|^{p-2} u}{\|u\|_p^{p-1}}, \quad u \in W_0^{1,p}(\Omega)$$

that has been normalized to be homogeneous of degree 0. The estimate

$$C_1 |\Omega| \lambda^n \leq \#\{j : \lambda_j < \lambda\} \leq C_2 |\Omega| \lambda^n \quad \text{for } \lambda > 0 \text{ large}$$

that Theorem 1.1 gives for the eigenvalues of this problem should be compared with the estimate

$$C_1 |\Omega| \lambda^{n/p} \leq \#\{j : \lambda_j < \lambda\} \leq C_2 |\Omega| \lambda^{n/p} \quad \text{for } \lambda > 0 \text{ large}$$

obtained by Friedlander in [6] for (1.4) (see also García Azorero and Peral Alonso [7]). Caliri and Squassina [1] have recently developed a numerical method to compute the first eigenpair of the problem (1.1) and investigate the symmetry breaking phenomena with respect to the constant case.

In the course of proving Theorem 1.1, we will also establish the same asymptotic estimates for the eigenvalues of the problem

$$-\operatorname{div} \left(\left| \frac{\nabla u}{L(u)} \right|^{p(x)-2} \frac{\nabla u}{L(u)} \right) = \mu T(u) \left| \frac{u}{l(u)} \right|^{p(x)-2} \frac{u}{l(u)}, \quad u \in W^{1,p(x)}(\Omega), \quad (1.5)$$

where

$$L(u) = \|\nabla u\|_{p(x)}, \quad l(u) = \|u\|_{p(x)}, \quad T(u) = \frac{\int_{\Omega} \left| \frac{\nabla u(x)}{L(u)} \right|^{p(x)} dx}{\int_{\Omega} \left| \frac{u(x)}{l(u)} \right|^{p(x)} dx}.$$

The eigenvalues and eigenfunctions of this problem on

$$\mathcal{N} = \{u \in W^{1,p(x)}(\Omega) : l(u) = 1\}$$

coincide with the critical values and critical points of $\tilde{L} := L|_{\mathcal{N}}$. Let \mathcal{G} denote the class of symmetric subsets of \mathcal{N} and set

$$\mu_j := \inf_{\substack{N \in \mathcal{G} \\ i(N) \geq j}} \sup_{u \in N} \tilde{L}(u), \quad j \geq 1.$$

Then (μ_j) is a sequence of eigenvalues of (1.5), $\mu_j \nearrow \infty$, and

$$i(\tilde{L}^\mu) = \#\{j : \mu_j < \mu\} \quad \forall \mu \in \mathbb{R},$$

where $\tilde{L}^\mu = \{u \in \mathcal{N} : \tilde{L}(u) < \mu\}$. Since $W^{1,p(x)}(\Omega) \supset W_0^{1,p(x)}(\Omega)$ and $l|_{W_0^{1,p(x)}(\Omega)} = k$, we have $\mathcal{N} \supset \mathcal{M}$, and $\tilde{L}|_{\mathcal{M}} = \tilde{K}$, so $\mu_j \leq \lambda_j$ for all j . We will see that, under the hypotheses of Theorem 1.1,

$$C_1 |\Omega| \mu^{n/(1+\sigma)} \leq \#\{j : \mu_j < \mu\} \leq C_2 |\Omega| \mu^{n/(1-\sigma)} \quad \text{for } \mu > 0 \text{ large.}$$

Finally, for the sake of completeness, let us also mention that a different notion of first eigenvalue for the $p(x)$ -Laplacian, that does not make use of the Luxemburg norm, has been considered in the past literature, namely,

$$\lambda_1^* = \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^{p(x)} dx}{\int_{\Omega} |u|^{p(x)} dx}.$$

In this framework, $\lambda \in \mathbb{R}$ and $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ are an eigenvalue and an eigenfunction of the $p(x)$ -Laplacian, respectively, if

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx = \lambda \int_{\Omega} |u|^{p(x)-2} uv dx \quad \forall v \in W_0^{1,p(x)}(\Omega)$$

(this should be compared with (1.1)). Let Λ denote the set of all eigenvalues of this problem. If the function $p(x)$ is a constant $p > 1$, then it is well-known that this problem admits an increasing sequence of eigenvalues, $\sup \Lambda = +\infty$, and $\inf \Lambda = \lambda_{1,p} > 0$, the first eigenvalue of the p -Laplacian (see Lindqvist [8, 9, 10]). For general $p(x)$, Λ is a nonempty infinite set, $\sup \Lambda = +\infty$, and $\inf \Lambda = \lambda_1^*$ (see Fan et al. [4]). In contrast to the situation when minimizing the Rayleigh quotient with respect to the Luxemburg norm, one often has $\lambda_1^* = 0$, and $\lambda_1^* > 0$ only under special conditions. In [4], the authors provide sufficient conditions for λ_1^* to be zero or positive. In particular, if $p(x)$ has a strict local minimum (or maximum) in Ω , then $\lambda_1^* = 0$. If $n > 1$ and there is a vector $\ell \neq 0$ in \mathbb{R}^n such that for every $x \in \Omega$, the map $t \mapsto p(x + t\ell)$ is monotone on $\{t : x + t\ell \in \Omega\}$, then $\lambda_1^* > 0$. Finally, if $n = 1$, then $\lambda_1^* > 0$ if and only if the function $p(x)$ is monotone.

2 Compactness

In this section we will show that \tilde{K} satisfies the (PS) condition. Here and in the sequel we will make use of the well-known Young's inequality

$$ab \leq \left(1 - \frac{1}{p}\right) a^{p/(p-1)} + \frac{1}{p} b^p \quad \forall a, b \geq 0, p > 1. \quad (2.1)$$

Lemma 2.1. *For $u \neq 0$ in $L^{p(x)}(\Omega)$ and all $v \in L^{p(x)}(\Omega)$,*

$$|(k'(u), v)| \leq \|v\|_{p(x)}. \quad (2.2)$$

Proof. Equality holds in (2.2) if $v = 0$, so suppose $v \neq 0$. We have

$$|(k'(u), v)| \leq \frac{\int_{\Omega} \left| \frac{u(x)}{k(u)} \right|^{p(x)-1} |v(x)| dx}{\int_{\Omega} \left| \frac{u(x)}{k(u)} \right|^{p(x)} dx}. \quad (2.3)$$

Taking $a = |u(x)/k(u)|^{p(x)-1}$, $b = |v(x)/k(v)|$, $p = p(x)$ in (2.1) and integrating over Ω gives

$$\int_{\Omega} \left| \frac{u(x)}{k(u)} \right|^{p(x)-1} \left| \frac{v(x)}{k(v)} \right| dx \leq \int_{\Omega} \left| \frac{u(x)}{k(u)} \right|^{p(x)} dx - \int_{\Omega} \left| \frac{u(x)}{k(u)} \right|^{p(x)} \frac{dx}{p(x)} + \int_{\Omega} \left| \frac{v(x)}{k(v)} \right|^{p(x)} \frac{dx}{p(x)}.$$

The last two integrals are both equal to 1, so this shows that the right-hand side of (2.3) is less than or equal to $k(v) = \|v\|_{p(x)}$. \square

Lemma 2.2. *K' is a mapping of type (S_+) , i.e., if $u_j \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and*

$$\overline{\lim}_{j \rightarrow \infty} (K'(u_j), u_j - u) \leq 0,$$

then $u_j \rightarrow u$ in $W_0^{1,p(x)}(\Omega)$.

Proof. Since

$$(K'(u_j), u_j) = K(u_j) = \|\nabla u_j\|_{p(x)}$$

and

$$(K'(u_j), u) = (k'(\nabla u_j), \nabla u) \leq \|\nabla u\|_{p(x)}$$

by Lemma 2.1,

$$\overline{\lim}_{j \rightarrow \infty} \|\nabla u_j\|_{p(x)} \leq \overline{\lim}_{j \rightarrow \infty} (K'(u_j), u_j - u) + \|\nabla u\|_{p(x)} \leq \|\nabla u\|_{p(x)} \leq \underline{\lim}_{j \rightarrow \infty} \|\nabla u_j\|_{p(x)},$$

so that $\|\nabla u_j\|_{p(x)} \rightarrow \|\nabla u\|_{p(x)}$. The conclusion follows since $W_0^{1,p(x)}(\Omega)$ is uniformly convex. \square

Lemma 2.3. *For all $c \in \mathbb{R}$, \tilde{K} satisfies the $(PS)_c$ condition, i.e., every sequence $(u_j) \subset \mathcal{M}$ such that $\tilde{K}(u_j) \rightarrow c$ and $\tilde{K}'(u_j) \rightarrow 0$ has a convergent subsequence.*

Proof. We have

$$K(u_j) \rightarrow c, \quad K'(u_j) - c_j k'(u_j) \rightarrow 0 \quad (2.4)$$

for some sequence $(c_j) \subset \mathbb{R}$. Since $(K'(u_j), u_j) = K(u_j)$ and $(k'(u_j), u_j) = k(u_j) = 1$, $c_j \rightarrow c$. Since (u_j) is bounded in $W_0^{1,p(x)}(\Omega)$, for a renamed subsequence and some $u \in W_0^{1,p(x)}(\Omega)$, $u_j \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $u_j \rightarrow u$ in $L^{p(x)}(\Omega)$. By Lemma 2.1,

$$|(k'(u_j), u_j - u)| \leq \|u_j - u\|_{p(x)} \rightarrow 0,$$

so the second limit in (2.4) now gives $(K'(u_j), u_j - u) \rightarrow 0$ as $j \rightarrow \infty$. Then we conclude that $u_j \rightarrow u$ strongly in $W_0^{1,p(x)}(\Omega)$, in light of Lemma 2.2. \square

3 Preliminaries on the genus and the cogenus

Recall that the genus and the cogenus of $M \in \mathcal{F}$ are defined by

$$\gamma(M) = \inf \{m \geq 1 : \exists \text{ an odd continuous map } g : M \rightarrow S^{m-1}\}$$

and

$$\tilde{\gamma}(M) = \sup \{\tilde{m} \geq 1 : \exists \text{ an odd continuous map } \tilde{g} : S^{\tilde{m}-1} \rightarrow M\},$$

respectively. If there are odd continuous maps $S^{\tilde{m}-1} \rightarrow M \rightarrow S^{m-1}$, then $\tilde{m} \leq i(M) \leq m$ by the monotonicity of the index, so $\tilde{\gamma}(M) \leq i(M) \leq \gamma(M)$. Since $\tilde{K}^\lambda \subset \tilde{L}^\lambda$, this gives

$$\tilde{\gamma}(\tilde{K}^\lambda) \leq i(\tilde{K}^\lambda) \leq i(\tilde{L}^\lambda) \leq \gamma(\tilde{L}^\lambda) \quad \forall \lambda \in \mathbb{R}. \quad (3.1)$$

Lemma 3.1. *If W_1 and W_2 are Banach spaces, M_1 and M_2 are symmetric subsets of $W_1 \setminus \{0\}$ and $W_2 \setminus \{0\}$, respectively, and*

$$M = \{(1-t)u_1, tu_2) \in W_1 \oplus W_2 : u_1 \in M_1, u_2 \in M_2, t \in [0, 1]\},$$

then

$$\tilde{\gamma}(M_1) + \tilde{\gamma}(M_2) \leq \tilde{\gamma}(M), \quad \gamma(M) \leq \gamma(M_1) + \gamma(M_2).$$

Proof. Since M contains copies of M_1 and M_2 , if $\tilde{\gamma}(M_1)$ or $\tilde{\gamma}(M_2)$ is infinite, then so is $\tilde{\gamma}(M)$ by monotonicity and hence the first inequality holds. So let $\tilde{m}_i := \tilde{\gamma}(M_i) < \infty$ and let $\tilde{g}_i : S^{\tilde{m}_i-1} \rightarrow M_i$ be an odd continuous map for $i = 1, 2$. Write $x \in S^{\tilde{m}_1+\tilde{m}_2-1}$ as $x = (x_1, x_2) \in \mathbb{R}^{\tilde{m}_1} \oplus \mathbb{R}^{\tilde{m}_2}$ and write $t = |x_2|$. Then

$$\tilde{g}(x) = \begin{cases} (\tilde{g}_1(x_1), 0), & t = 0 \\ ((1-t)\tilde{g}_1(x_1/\sqrt{1-t^2}), t\tilde{g}_2(x_2/t)), & 0 < t < 1 \\ (0, \tilde{g}_2(x_2)), & t = 1 \end{cases}$$

defines an odd continuous map $\tilde{g} : S^{\tilde{m}_1+\tilde{m}_2-1} \rightarrow M$ and hence $\tilde{\gamma}(M) \geq \tilde{m}_1 + \tilde{m}_2$.

Since the second inequality holds if $\gamma(M_1)$ or $\gamma(M_2)$ is infinite, let $m_i := \gamma(M_i) < \infty$ and let $g_i : M_i \rightarrow S^{m_i-1}$ be an odd continuous map for $i = 1, 2$. Then

$$g(((1-t)u_1, tu_2)) = (\sqrt{1-t^2}g_1(u_1), tg_2(u_2)), \quad u_1 \in M_1, u_2 \in M_2, t \in [0, 1]$$

defines an odd continuous map $g : M \rightarrow S^{m_1+m_2-1}$ and hence $\gamma(M) \leq m_1 + m_2$. \square

4 Some lemmas

Lemma 4.1. *If Ω_1 and Ω_2 are disjoint subdomains of Ω such that $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$, then*

$$\tilde{\gamma}(\tilde{K}_{\Omega_1}^\lambda) + \tilde{\gamma}(\tilde{K}_{\Omega_2}^\lambda) \leq \tilde{\gamma}(\tilde{K}_\Omega^\lambda), \quad \gamma(\tilde{L}_\Omega^\lambda) \leq \gamma(\tilde{L}_{\Omega_1}^\lambda) + \gamma(\tilde{L}_{\Omega_2}^\lambda) \quad \forall \lambda \in \mathbb{R},$$

where the subscripts indicate the corresponding domains.

Proof. We have

$$\{((1-t)u_1, tu_2) \in W_0^{1,p(x)}(\Omega_1) \oplus W_0^{1,p(x)}(\Omega_2) : u_1 \in \tilde{K}_{\Omega_1}^\lambda, u_2 \in \tilde{K}_{\Omega_2}^\lambda, t \in [0, 1]\} \subset \tilde{K}_\Omega^\lambda$$

and

$$\tilde{L}_\Omega^\lambda \subset \{((1-t)u_1, tu_2) \in W^{1,p(x)}(\Omega_1) \oplus W^{1,p(x)}(\Omega_2) : u_1 \in \tilde{L}_{\Omega_1}^\lambda, u_2 \in \tilde{L}_{\Omega_2}^\lambda, t \in [0, 1]\},$$

so the conclusion follows from Lemma 3.1 and the monotonicity of the cogenus and the genus. \square

Lemma 4.2. *Let $0 < \tau < 1$, consider the homothety $\Omega \rightarrow \tau\Omega$, $x \mapsto \tau x =: y$, and write $p(x) = q(y)$ and $u(x) = v(y)$. If $1 < p^- \leq p(x) \leq p^+ < \infty$ for all $x \in \Omega$ and $u \neq 0$ is in $W^{1,p(x)}(\Omega)$, then*

$$\tau^{\sigma-1} \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \leq \frac{\|\nabla v\|_{q(y)}}{\|v\|_{q(y)}} \leq \tau^{-\sigma-1} \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}},$$

where $\sigma = n(1/p^- - 1/p^+)$.

Proof. We have

$$\int_{\tau\Omega} \left| \frac{v(y)}{\nu} \right|^{q(y)} \frac{dy}{q(y)} = \int_{\Omega} \tau^n \left| \frac{u(x)}{\nu} \right|^{p(x)} \frac{dx}{p(x)} = \int_{\Omega} \left| \frac{\tau^{n/p(x)} u(x)}{\nu} \right|^{p(x)} \frac{dx}{p(x)},$$

so $\|v\|_{q(y)} = \|\tau^{n/p(x)} u\|_{p(x)}$. Since $\tau < 1$, $\tau^{n/p^-} \leq \tau^{n/p(x)} \leq \tau^{n/p^+}$, so it follows that

$$\tau^{n/p^-} \|u\|_{p(x)} \leq \|v\|_{q(y)} \leq \tau^{n/p^+} \|u\|_{p(x)}.$$

Similarly, we obtain

$$\tau^{n/p^- - 1} \|\nabla u\|_{p(x)} \leq \|\nabla v\|_{q(y)} \leq \tau^{n/p^+ - 1} \|\nabla u\|_{p(x)}$$

since $\nabla v(y) = \nabla u(x)/\tau$. The conclusion follows. \square

Lemma 4.3. *If $1 < p^- \leq p(x) \leq p^+ < \infty$ for all $x \in \Omega$ and $(1/p^- - 1/p^+) |\Omega| < 1$, then*

$$C^- \|u\|_{p^-} \leq \|u\|_{p(x)} \leq C^+ \|u\|_{p^+} \quad \forall u \in L^{p^+}(\Omega), \quad (4.1)$$

where

$$C^- = \left[1 + \left(\frac{1}{p^-} - \frac{1}{p^+} \right) |\Omega| \right]^{-1/p^-}, \quad C^+ = \left[1 - \left(\frac{1}{p^-} - \frac{1}{p^+} \right) |\Omega| \right]^{-1/p^+},$$

and hence

$$\frac{1}{C} \frac{\|\nabla u\|_{p^-}}{\|u\|_{p^+}} \leq \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \leq C \frac{\|\nabla u\|_{p^+}}{\|u\|_{p^-}} \quad \forall u \in W^{1,p^+}(\Omega) \setminus \{0\},$$

where $C = C^+/C^-$.

Proof. Equality holds throughout (4.1) if $u = 0$, so suppose $u \neq 0$. Taking $a = 1$, $b = |u(x)/\|u\|_{p(x)}|^{p^-}$, $p = p(x)/p^-$ in (2.1), dividing by p^- , and integrating over Ω gives

$$\frac{1}{\|u\|_{p(x)}^{p^-}} \int_{\Omega} |u(x)|^{p^-} \frac{dx}{p^-} \leq \int_{\Omega} \left(\frac{1}{p^-} - \frac{1}{p(x)} \right) dx + \int_{\Omega} \left| \frac{u(x)}{\|u\|_{p(x)}} \right|^{p(x)} \frac{dx}{p(x)}.$$

The first integral is equal to $\|u\|_{p^-}^{p^-}$ and the last integral is equal to 1, so this gives the first inequality in (4.1). Now taking $a = 1$, $b = |u(x)/\|u\|_{p(x)}|^{p(x)}$, $p = p^+/p(x)$ in (2.1), dividing by $p(x)$, and integrating over Ω gives

$$\int_{\Omega} \left| \frac{u(x)}{\|u\|_{p(x)}} \right|^{p(x)} \frac{dx}{p(x)} \leq \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) dx + \frac{1}{\|u\|_{p(x)}^{p^+}} \int_{\Omega} |u(x)|^{p^+} \frac{dx}{p^+}.$$

The first integral is equal to 1 and the last integral is equal to $\|u\|_{p^+}^{p^+}$, so this gives the second inequality in (4.1). \square

Let us set

$$\widehat{K}(u) := \|\nabla u\|_{p^+}, \quad u \in \widehat{\mathcal{M}} := \{u \in W_0^{1,p^+}(\Omega) : \|u\|_{p^-} = 1\}$$

and

$$\widehat{L}(u) := \|\nabla u\|_{p^-}, \quad u \in \widehat{\mathcal{N}} := \{u \in W^{1,p^+}(\Omega) : \|u\|_{p^+} = 1\}.$$

Lemma 4.4. *If $0 < \tau < 1$, $1 < p^- \leq p(x) \leq p^+ < \infty$ for all $x \in \Omega$, $\sigma = n(1/p^- - 1/p^+)$, $(1/p^- - 1/p^+)|\Omega| < 1$, and C is as in Lemma 4.3, then*

$$\widetilde{\gamma}(\widehat{K}_{\Omega}^{\lambda}) \leq \widetilde{\gamma}(\widetilde{K}_{\tau\Omega}^C \tau^{-\sigma-1\lambda}), \quad \gamma(\widetilde{L}_{\tau\Omega}^{\tau\sigma-1\lambda/C}) \leq \gamma(\widehat{L}_{\Omega}^{\lambda}) \quad \forall \lambda \in \mathbb{R}.$$

Proof. Lemmas 4.2 and 4.3 give the odd continuous maps

$$\widehat{K}_{\Omega}^{\lambda} \rightarrow \widetilde{K}_{\tau\Omega}^C \tau^{-\sigma-1\lambda}, \quad u \mapsto \frac{v}{\|v\|_{q(y)}}, \quad \widetilde{L}_{\tau\Omega}^{\tau\sigma-1\lambda/C} \rightarrow \widehat{L}_{\Omega}^{\lambda}, \quad v \mapsto \frac{u}{\|u\|_{p^+}},$$

so the conclusion follows from monotonicity. \square

5 Proof of Theorem 1.1

By using the Tietze extension theorem, we can continuously extend p to the whole space, with the same bounds p^- and p^+ . Fix $a_0 > 0$ so small that $(1/p^- - 1/p^+)a_0^n < 1$, let Q_{a_0} be a cube of side a_0 , fix $\lambda_0 > \max\{\inf \widehat{K}_{Q_{a_0}}, \inf \widehat{L}_{Q_{a_0}}\}$, and set

$$r = \widetilde{\gamma}(\widehat{K}_{Q_{a_0}}^{\lambda_0}), \quad s = \gamma(\widehat{L}_{Q_{a_0}}^{\lambda_0}).$$

Then for $\lambda > C\lambda_0$, where $C > 1$ is as in Lemma 4.3 applied to Q_{a_0} , and for any two cubes $Q_{a_{\lambda}}$ and $Q_{b_{\lambda}}$ of sides $a_{\lambda} = (C\lambda_0/\lambda)^{1/(1+\sigma)}a_0$ and $b_{\lambda} = (\lambda_0/C\lambda)^{1/(1-\sigma)}a_0$, respectively,

$$r \leq \widetilde{\gamma}(\widetilde{K}_{Q_{a_{\lambda}}}^{\lambda}), \quad \gamma(\widetilde{L}_{Q_{b_{\lambda}}}^{\lambda}) \leq s$$

by Lemma 4.4. Now it follows from Lemma 4.1 that if Q_a is a cube of side $a > 0$, then

$$r \left[\frac{a}{a_{\lambda}} \right]^n \leq \widetilde{\gamma}(\widetilde{K}_{Q_a}^{\lambda}), \quad \gamma(\widetilde{L}_{Q_a}^{\lambda}) \leq s \left(\left[\frac{a}{b_{\lambda}} \right] + 1 \right)^n,$$

where $[\cdot]$ denotes the integer part. Thus, there are constants $C_1, C_2 > 0$, independent of a and λ , such that

$$C_1 a^n \lambda^{n/(1+\sigma)} \leq \tilde{\gamma}(\tilde{K}_{Q_a}^\lambda), \quad \gamma(\tilde{L}_{Q_a}^\lambda) \leq C_2 a^n \lambda^{n/(1-\sigma)}, \quad \lambda \text{ large.} \quad (5.1)$$

Let $\varepsilon > 0$ and let $\Omega_\varepsilon, \Omega^\varepsilon$ be unions of cubes with pairwise disjoint interiors such that $\Omega_\varepsilon \subset \Omega \subset \Omega^\varepsilon$ and $|\Omega^\varepsilon \setminus \Omega_\varepsilon| < \varepsilon$. Then

$$C_1 |\Omega_\varepsilon| \lambda^{n/(1+\sigma)} \leq \tilde{\gamma}(\tilde{K}_{\Omega_\varepsilon}^\lambda) \leq \tilde{\gamma}(\tilde{K}_\Omega^\lambda), \quad \gamma(\tilde{L}_\Omega^\lambda) \leq \gamma(\tilde{L}_{\Omega^\varepsilon}^\lambda) \leq C_2 |\Omega^\varepsilon| \lambda^{n/(1-\sigma)}, \quad \lambda \text{ large}$$

by (5.1). Letting $\varepsilon \rightarrow 0$ and combining with (3.1) and (1.3) yields the conclusion.

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